Documenting the Knowledge of Low-Attaining Third- and Fourth-Graders: Robyn's and Bel's Sequential Structure and Multidigit Addition and Subtraction

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Aspects of students' arithmetic knowledge are described via two case studies of responses to tasks during a videotaped assessment interview. Tasks include reading numerals, locating numbers, saying number word sequences by ones and tens, number word after or before a given number, incrementing and decrementing by ten, addition in the context of dot strips of tens and ones, and addition and subtraction involving bare numbers. On many tasks the students had significant difficulties and responded differently from each other. The paper demonstrates the idiosyncratic nature of arithmetical knowledge, and the significance of context in students' multidigit thinking.

The paper reports on aspects of a current 3-year project that has the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). These tools include schedules of diagnostic assessment tasks, and a learning framework for profiling students' number knowledge. A particular focus of study has been assessment of student knowledge of multidigit addition and subtraction. Most research on multidigit knowledge is with younger students' initial learning of multidigit arithmetic. For low-attaining older students, who may already have been expected to master 2-digit column algorithms, we wish to establish a profile of their multidigit knowledge. The paper describes two case study profiles.

Literature Review

In the last 15 years, research and curriculum reforms in a range of countries highlight a renewed emphasis on mental computation with multidigit numbers (Beishuizen & Anghileri, 1998; Cooper, Heirdsfield, & Irons, 1995; McIntosh, Reys, & Reys, 1992; Thompson & Smith, 1999). An emphasis on mental strategies may (a) support conceptual understanding of multidigit numbers (Fuson et al., 1997; Heirdsfield, 2005; Hiebert & Wearne, 1996); (b) support development of number sense and important connections to related knowledge (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997; McIntosh et al., 1992; Sowder, 1992); and (c) stimulate the development of numerical reasoning, and flexible, efficient computation (Beishuizen & Anghileri, 1998; Yackel, 2001). Following the principle of beginning instruction with students' informal strategies, researchers now put initial instructional emphasis on strong mental strategies (Beishuizen & Anghileri, 1998; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998).

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Place Value and Base-ten Structures

Multidigit knowledge includes knowledge of the numeration system and place value (e.g., Hiebert & Wearne, 1996). However, researchers argue that students may not operate with numbers in symbolic terms, observing that place value tasks become tasks of verbal patterns and symbolic manipulation, without connection to the students' sense of numbers (Cobb & Wheatley, 1988; Treffers, 1991). Thompson and Bramald (2002) make a distinction between quantity value, for example, partitioning 47 into forty and seven, and column value, for example, 47 represents 4 units of ten and 7 units of one. They argue students' mental strategies only depend on quantity value. In this paper we focus on base-ten structures that include aspects of place value knowledge, such as quantity value, which do not involve manipulating written symbols.

Of central interest in students' mental multidigit computation is the developing sophistication of their use of base-ten structures. Researchers have charted learning trajectories from using counting-by-ones strategies, through increasingly powerful uses of units of ten and other base-ten structures. In a synthesis from four research projects, Fuson et al. (1997) proposed a developmental sequence of children's two-digit conceptual structures. The structures incorporate students' relations among written numerals, number words, and quantities: *unitary* (53 as one, two, … fifty-three); *decade and ones* (one, two … fifty; and fifty-one, fifty-two, fifty-three); *sequence-tens and ones* (ten, twenty, … fifty; and fifty-one, fifty-two, fifty-three); *separate-tens and ones* (five tens and three ones); and *integrated-sequence-separate*. A sixth, incorrect conceptual structure was labelled *concatenated single digit* (53 as five and three). Developing the work of Steffe and colleagues, Cobb and Wheatley (1988) distinguished three levels in children's construction of ten as a unit. The levels were evident in children's thinking in additive tasks. Children operating at level 1 manipulate ten units and one units separately, and can not coordinate them. The level 1 construction of ten as an *abstract singleton* is comparable to the *concatenated single digit* structure from Fuson and colleagues. At level 2, children can coordinate counts or collections of tens and of ones, in the context of representations of the quantities, but they cannot "simultaneously construct a numerical whole and the units of ten and one that compose it" (p. 7). Students at level 3 can anticipate, without representations, that a numerical whole consists of tens and ones units, and coordinate operations with these. Significant in these analyses is the consideration of students' thinking in two settings: structured materials and bare numbers. The present study investigates students' use of base-ten structures and units when solving additive tasks in three settings: structured materials, bare numbers, and verbal number words.

Sequence-based Structure and Strategies

When students begin to use base-ten structures in arithmetic, they develop a variety of multidigit addition and subtraction strategies (Beishuizen & Anghileri, 1998; Cooper et al., 1995; Foxman & Beishuizen, 2002; Thompson & Smith, 1999). Sequence-based or jump strategies involve keeping the first number whole and adding (or subtracting) via a series of jumps, for example, 57 + 26 as 57 + 10, 67 + 10, 77 + 3, and 80 + 3. Collections-based or split strategies involve partitioning both numbers into tens and ones, and adding (or subtracting) separately with tens and ones, for example, 50 + 20, 7 + 6, and 70 + 13.

A broad knowledge of number relationships and numeration is important for mental computation (Heirdsfield, 2001). This includes knowledge of sequential structure

(Ellemor-Collins & Wright, in press): jumping by ten off the decade, locating numbers, number word sequences across decades, and making small hops (Fuson et al., 1997; Menne, 2001; Yackel, 2001).

Beishuizen and Anghileri (1998) argued that jump strategies can develop as curtailments of students' informal counting strategies. Beishuizen, Van Putten, and Van Mulken (1997) compared students' use of jump and split strategies and found that jump resulted in fewer errors and enabled making efficient computation choices. In contrast, split strategies led to difficulty in developing independence from concrete materials (Beishuizen, 1993); procedural and conceptual confusion (Klein, Beishuizen, & Treffers, 1998); and slow response times, suggesting a heavier load on working memory (Wolters, Beishuizen, Broers, & Knoppert, 1990). As well, Klein, Beishuizen, and Treffers (1998) found that, among low-attainers, jump strategies were much more successful.

Low-attaining students seem to use jump strategies less frequently and many do not develop knowledge of jumping in tens (Beishuizen, 1993; Foxman & Beishuizen, 2002; Menne, 2001). In Australia in many instances, instruction does not focus on counting by tens off the decade nor on developing sequential structure. Yet, sequence-based strategies can be more successful, and are necessary for integrating sequence-based and collections-based constructions (Fuson et al., 1997). Hence, the focus of this study is on low-attaining students' development of sequential structure and jump strategies.

Low-attaining Students

Students' arithmetic knowledge is componential (Dowker, 2005) and for students of similar ability levels, there can be significant differences in arithmetic knowledge profiles (Gervasoni, 2005). Understanding more about such profiles is one important response to calls for intervention in early number learning (Louden et al, 2000; Department of Education, Training and Youth Affairs, 2000). Further, assessment of students' multidigit knowledge should include a focus on multidigit numerals, number sequence knowledge, ten as a unit, mental computation, in verbal, structured, and bare number settings, and attention to students' strategies, as well as their answers. This paper presents two case studies that (a) describe in detail, low-attaining students' multidigit knowledge; (b) illustrate the idiosyncratic nature of this knowledge; and (c) illustrate the significance of context in students' multidigit thinking.

Method

Study

A screening test of arithmetical knowledge was administered to all third- and fourth-grade students in 17 schools. On the basis of the screening test, 191 students were classified as low-attaining. During their year in the project – 2004 or 2005 – these students were assessed twice, that is, in the second term and in the fourth term. The assessment consisted of an individual interview, videotaped for subsequent analysis. The analysis documents in detail each student's responses and strategies.

Task Groups

The interview used a schedule of task groups. A task group consists of tasks very similar to each other used to document students' knowledge of a specific topic. Some of these tasks are adapted from Cobb and Wheatley (1988) and have been widely used elsewhere (e.g., New South Wales Department of Education and Training, 2003). This paper focuses on eight of 20 task groups in the schedule:

1. *Numerals task group.* These tasks involved identifying and writing numerals. This included numerals with up to 5 digits and 3- and 4-digit numerals with a zero (e.g., 12, 21; 101, 730, 306; 1000, 1006, 3406, 6032, 3010; 10 235).
2. *Locating numbers task group*. Given a piece of paper showing a line with ends labelled as 0 and 100, the task was to mark in turn, 50, 25, 98, and 62.
3. *Number word sequences (NWS) by ones.* These tasks involved (a) saying a forward (FNWS) or backward (BNWS) sequence and included bridging decades, 100s, and 1000; and (b) saying the number before or after a given number.
4. *Number word sequences by tens.* Saying sequences by tens, forward or backward, in the range 1 to 1000, on and off the decade.
5. *Incrementing and decrementing using numerals.* Given a numeral, say the number that is ten more, using: 20, 90, 79, 356, 306, 195, and 999. Similarly, ten less than: 30, 79, 356, 306, 1005; one hundred more than: 50, 306, 973; one hundred less than 108.
6. *Incrementing and decrementing using ten-strips.* A strip with seven dots is placed out, then strips with ten dots are used one by one. The student's task is to state the total number after each successive strip is placed out – 7, 17, 27 etc.
7. *Incrementing using tens and ones.* Strips with the following numbers of dots are progressively uncovered: 4, one 10, two 10s, one 10 and 4, two 10s and 5. The student's task is to state the total number of dots at each successive uncovering. Finally, the 73 dots are covered and the student is asked how many more dots are needed to make 100.
8. *Bare number tasks.* The following are presented in horizontal format for the student to solve without materials or paper for writing: 43 + 21, 37 + 19, 86 – 24, 50 – 27.

Results

The case studies in this paper are based on the first interviews of two students. Of particular interest in the case of Bel are (a) his inability to jump by ten off the decade, in the absence of materials; and (b) his difficulties with addition and subtraction tasks requiring regrouping. Of particular interest in the case of Robyn are (a) her facility with jumping by ten off the decade, and (b) her difficulties with addition and subtraction tasks in bare number settings.

The Case of Bel

Bel was 9 years and 4 months old at the time of his interview, 15 weeks into the third grade (fourth year of school).

Numerals and locating numbers. Bel wrote correctly, all 3- and 4-digit numerals asked (270, 306, 1000, 1005, 2020), and identified all 3-digit numerals (101, 400, 275, 730, 306) and all but one of the 4-digit numerals (1000, 8245, 1006, 3406, 6032, 1300). His error was to identify 3010 as “three hundred and ten”. Bel’s location for 50 on the number line from 0 to 100 was quite accurate. His locations for 25, 62, and 98 were correctly ordered but inaccurate.

Number word sequences. Bel recited four FNWSs and BNWSs in the range 1 to 120. This included two self-corrections. He recited the BNWS from 303 but could not continue beyond 298. As well, he was partially successful with sequences bridging 1000. He recited the sequence from 1010 to 995, but made errors as follows: “1003, 1002, 1001, 999, 998” and “993, 992, 991, 990, 899, 888”. He was successful on nine number word after tasks and ten number word before tasks in the range one to 2000. He made one error only on this kind of task, that is, he stated “seven hundred and sixty-nine” as the number before 170.

Number word sequences by ten and incrementing by ten. Bel recited the sequence of decuples from 10 to 120 forward and backward, and other sequences of decuples up to 1090 but he could not count by tens from 24. As well, he could increment and decrement by 10 on the decade but not off the decade. His errors were to answer “81” as 10 more than 79, “315” as 10 more than 356, “61” as 10 less than 79, and “259” as 10 less than 356. By contrast, he correctly stated 100 more than 306, 100 more than 973, and 100 less than 108. In the context of ten-strips, Bel incremented by 10 off the decade – “27, 37, 47…”, but appeared to count by ones from seven, to figure out 7 dots plus 10 dots.

Incrementing using tens and ones. Bel was partially successful on the task involving strips and incrementing using tens and ones. He incremented 34 by 14, and in doing so, appeared to use a split-jump strategy, that is, 30 + 10, 40 + 4 and 44 + 4, counting by ones to figure out 44 + 4. In attempting to increment 48 by 25, he answered “33” after 43 seconds. When asked to explain, he pointed to each of the two ten-strips in turn, in coordination with saying “58, 68”. He then counted by ones as follows: “69, 30, 31, 32, 33”. He apparently used a jump strategy but could not correctly keep track when counting by ones from 68. Note that (a) Bel used a relatively low-level strategy, that is counting on by ones, to figure out 44 and 4, and 68 and 5. In both cases the items to count were perceptually available. (b) In the context of ten-strips, he incremented 48 by two tens, but (as described earlier), on a verbal task he could not count by tens from 24 and could not state 10 more than 79.

Bare number tasks. Bel used a split strategy to solve each of 43 + 21 and 86 – 24. For 37 + 19 he answered “68”. According to his explanation, he first added 3 and 1. These solutions contrasted with his jump strategy in the context of ten-strips, for incrementing 48 by 25 (as described earlier). For 50 – 27 he answered “28”. According to his explanation, “I took away 2 off that”, indicating the 5 of 50, “then when I got down to 30, I took away 7”.

The Case of Robyn

Robyn was 9 years and 5 months old at the time of her interview, 15 weeks into the fourth-grade (fifth year of school).

Numerals and locating number. Robyn showed fluency with 3-digit numerals, and made three errors with 4-digit numerals. She correctly wrote 270, 306, 1000, 1005, and 4320. When asked to write “one thousand nine-hundred” she wrote “1009”. She correctly identified 101, 400, 275, 730, 306, 1000, 8245, 1006, 3406, 3010; she identified 6032 as “six hundred and thirty-two”, and then corrected herself, and identified 1300 as “thirteen thousand". In the locating numbers task, Robyn placed 50 correctly but, like Bel, her marks to locate 25, 62, and 98 were correctly ordered but inaccurate.

Number word sequences. In the range 1 to 1000, Robyn recited eight number word sequences, and stated the number word before or after for twenty-five given numbers. She made five errors across these tasks, each of which she promptly, spontaneously corrected. Sequences across 1000 and beyond were problematic for Robyn, which we detail further below.

Number word sequences by ten and incrementing by ten. Robyn counted by tens on and off the decade, up to 120. With sequences beyond 120, she had difficulties bridging hundreds saying "170, 180, 190, 800, 810 …", and "177, 187, 197" (pause), "207" pause, "227, 237". Robyn successfully incremented and decremented by ten from on and off the decade in the range to 1000. She was fluent with eight such tasks, but she had significant difficulty with the task of incrementing 195 by ten and her response was indiscernible. Robyn was more successful on these tasks than many of the other low-attaining students. By contrast, Robyn could not increment by one hundred off the hundred: For 100 more than 50 she answered, "five hundred", and for 100 more than 306 she answered, "4006 … 406 … 4006".

Sequences across 1000. Robyn was unsuccessful with tasks that involved bridging 1000, apart from correctly stating the number word before 1000 and after 1000. She stated the forward sequence by ones as, "997, 998, 999, ten hundred, ten thousand (pause), ten hundred and one, ten hundred and two", and the backward sequence by ones as, "1002, 1001, 1000, nine-, 999, 989, 998 (as a correction for 989), 997, 996". For the forward sequence by tens she said, "970, 980, 990, 10 000, 10 010, 10 020", and for the forward sequence by hundreds she said, "800, 900 (six-second pause), 1000, 2000, 300, 3000 (as a correction for 300)". For the task of incrementing 999 by 10, she said "10 009", and for the task of decrementing 1005 by 10, she said "905".

Incrementing using tens and ones. On the task with 48 covered, and two ten-strips and five dots uncovered, Robyn counted subvocally, "48, 58, 68, 69 (pause), 70, 71, 72, 73", that is, she used a jump strategy that involved jumping two tens and counting by ones. Robyn was then asked how many more dots (from 73) would be needed to make 100. She made four attempts to solve this task and all of her attempts were unsuccessful. On the first three attempts her strategy was to count by ones from 73, and keep track of her counts on her fingers, but she seemed to lose track after about ten counts. Her fourth attempt appeared to involve a different strategy. She thought for 30 seconds in conjunction with some finger movements, and then answered "906". Thus Robyn was able to count in tens on the task involving addition with strips but not on the missing addend task.

Bare number tasks. Robyn did not solve successfully the three bare number tasks that were presented to her. For 43 + 21, she answered "604", and for part of her solution she counted by ones using her fingers to keep track. For 37 + 19, she answered "406" and for 86 – 24, she answered "994". On all three problems, Robyn appeared to use a split strategy and to recombine the tens and ones unsuccessfully. She apparently did not assess the appropriateness of her answers.

Discussion

Table 1 sets out descriptions of Bel's and Robyn's responses to numeral identification tasks, sequential structure tasks, and additive tasks. On the sequential structure tasks Bel's and Robyn's responses were significantly different from each other. This suggests that students' learning of topics related to sequential structure such as incrementing by ten or 100 on and off the decade and extending this to beyond 1000 can progress in different ways. Robyn's proficiency with jumping by ten off the decade contrasted significantly with Bel's lack of proficiency. However, Robyn did not use jumping by ten on the bare number tasks. Rather, she used split strategies. As well, on the addition task with ten-strips, Robyn was not more proficient than Bel.

Table 1
Summary Descriptions of Bel's and Robyn's Responses to Assessment Tasks

Task	Bel's response	Robyn's response
Numerals	Successful on all but one 4-digit task	Successful for 3-digit numerals
	Sequential structure tasks	
Locating numbers	Correct order but not accurate	Correct order but not accurate
NWS	Five errors	No errors, four self-corrections
NWS by ten: on decade	Successful to 1000	Successful to 120
NWS by ten: off decade	Unsuccessful	Successful to 120
Increment by ten	Unsuccessful off the decade	Successful to 1000
Increment by 100	Successful to 1000	Unsuccessful
Sequences across 1000	All four correct	Unsuccessful
	Additive tasks	
Ten-strips: 48+25	Jump strategy, could not keep track	Jump strategy
Ten-strips: 73+☐=100	Not assessed	Unsuccessful
Written: 43+21, 86 – 24	Split strategy	Split strategy, unsuccessful
Written: 37+19, 50 – 27	Different strategies, unsuccessful	Split strategy, unsuccessful

Bel's and Robyn's solutions to additive tasks indicate, in different ways, knowledge of the base-ten structure of numbers. On tasks involving ten-strips they used jump strategies and were partially successful. Their coordination of tens and ones suggests a sequence-tens and ones conception (Fuson et al., 1997), and a construction of at least a level 2 unit of ten (Cobb & Wheatley, 1988). Robyn's inability to construct a solution to the subsequent missing addend task suggests she had not yet constructed a level 3 unit of ten. On bare number tasks Bel and Robyn used split-based strategies and were less successful. Bel's different approaches to 37 + 19 and 50 – 27 suggest an integrated-sequence-separate conception. Robyn's responses suggest a concatenated single-digit conception of the written numbers, using only a level 1 unit of ten. Cobb and Wheatley (1988) also observed differences in students' responses to bare number tasks compared with tasks involving ten-strips.

On the additive task of 48 and 25 involving ten-strips, both Bel and Robyn counted by ones to add 68 and 5, and these solutions seemed to require significant effort. Bel counted by ones to add 44 and 4 involving ten-strips, even though elsewhere in the interview he solved 4 + 4 immediately (without counting by ones). Also, in the bare number tasks, Bel made errors adding 7 to 9 for 37 + 19, and subtracting 7 from 30 for 50 – 27. Further, in solving addition and subtraction problems in the range 1 to 20 (not described in the above case studies), both students used counting by ones and had difficulties. Thus Bel and Robyn lacked facility with addition and subtraction in the range 1 to 20 and, when doing addition and subtraction in the range 1 to 100, were not able to apply facts in the range 1 to 20 that they had habituated.

Some researchers have linked low-attainers' difficulties such as those described above, with broader aspects of their thinking. Drawing on Gray and Tall (1994), we observe that Robyn and Bel tended to use procedural thinking, which involves counting by ones and splitting, rather than proceptual thinking which involves for example, using 4 + 4 to work out 44 + 4, and coordinating units. Nevertheless, the students' use of jump strategies on the tens-strips tasks seemed to be more appropriate than their use of split strategies on the bare number tasks. Because of this, we contend that their difficulties can be attributed in part to confronting numbers in settings that do not yet make sense to them (Cobb & Wheatley, 1988). Drawing on analyses of mathematical development (Thomas, Mulligan, & Goldin, 2002), we contend that Robyn's and Bel's weak sense of locating numbers indicate low levels of knowledge of mathematical structure, which is linked with low-attainment.

Conclusions

As shown in the two case studies, the process of documenting a student's current arithmetical knowledge in terms of the eight aspects addressed in this study, highlights the complexities of that knowledge and its idiosyncratic nature (Gervasoni, 2005). Students' knowledge of the sequential structure of multi-digit numbers can be regarded as somewhat distinct from their place value knowledge. This refers to place value knowledge in a collections-based sense (Yackel, 2001). We contend that developing in students a rich knowledge of sequential structure is important and can provide an important basis for the development of mental computation.

The case studies confirm that facility with addition and subtraction involving a 1-digit number is a significant aspect of facility with 2-digit calculation (Heirdsfield, 2001). We contend that low-attainers need to develop their facility with 1-digit numbers in order to develop efficient strategies for multidigit calculations. Also confirmed in the case studies, is that students can learn to read and write numerals well in advance of learning place value in a collections-based sense (Wright, 1998). For this reason, we advocate that assessment frameworks should treat numeral identification (reading numerals) and place value (interpreting numerals) as separate domains of knowledge.

As well, the case studies illustrate that a student's mental strategies and number sense can differ from, on one hand, a context involving base-ten materials to, on the other hand, tasks based on bare numbers. This accords with the finding by Cobb and Wheatley (1988) that "the horizontal sentences and tens tasks were separate contexts for the children. The meanings that they gave to two-digit numerals or number words in the two situations were unrelated" (p.18). Related to this, students' strategies for addition and subtraction in bare number contexts can be relatively unsophisticated. Therefore low-attaining students are likely to need explicit instruction in order to extend their multi-digit number sense from contexts involving materials to contexts involving written arithmetic (Beishuizen & Anghileri, 1998; Heirdsfield, 2005; Treffers & Buys, 2001). Finally, the case studies demonstrate the use of assessment tasks to document students' knowledge and that the assessment should include (a) tasks involving base-ten materials, (b) verbally-based tasks, and (c) bare number tasks.

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